

HJÄLPMEDEL: Utdelat formelblad.

Lösningarna ska vara försedda med ordentliga motiveringar.

1. Låt talföljden $(a_n)_{n \in \mathbb{N}}$ definieras genom

$$a_{n+2} - 3a_{n+1} + 2a_n = 2^n, \quad a_0 = 1, a_1 = 1.$$

a) Bestäm lösningen till rekursionsekvationen. (0.8)

b) Konvergerar serien $\sum_{n=0}^{\infty} \left(\frac{a_n}{a_{n+1}}\right)^n$? (0.2)

2. a) Bestäm alla (komplexa) lösningar till ekvationen $\cos(z + \frac{3}{2}\pi) = -2$. (0.7)

b) Om man utvecklar funktionen $f(z) = \frac{1}{\cos(z + \frac{3}{2}\pi) + 2}$ i en potensserie $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, vilken konvergensskiva har den potensserien? (Du behöver inte beräkna koefficienterna a_n !) (0.3)

3.a) Beräkna den exponentiella och trigonometriska Fourierserien för den 2π -periodiska funktionen $f(t) = |\sin t|$ och rita f i intervallet $[-\pi, \pi]$. (0.7)

b) Beräkna seriesumman $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2}$. (0.3)

4. Låt $f(x+iy) = u(x, y) + iv(x, y)$ vara en hel funktion. Antag att $u(x, y) = g(x) \cos y$ för en två gånger kontinuerligt deriverbar funktion g . Bestäm $f(z)$ som uppfyller $f(0) = 1 + i$ och $f(i\frac{\pi}{2}) = 2i$.

5. a) Beräkna $\int_0^\infty \frac{\cos(2x)}{1+x^2} dx$. (0.6)

b) Låt γ vara en sluten slät kurva i $\mathbb{C} \setminus \{\pm i\}$. Vilka värden kan integralen $\int_\gamma \frac{\cos(2z)}{1+z^2} dz$ anta? (0.4)

6. a) För vilka komplexa tal z konvergerar/divergerar serien $\sum_{n=1}^\infty \frac{z^n}{1-z^n}$? (0.7)

b) För ett naturligt tal n defineras funktionsvärdet $\psi(n)$ som antalet positiva delare till n . Exempelvis delar 1, 2, 3, 6 talet 6 och $\psi(6) = 4$. Visa att

$$\sum_{n=1}^\infty \frac{z^n}{1-z^n} = \sum_{n=1}^\infty \psi(n) z^n$$

för $|z| < 1$. (0.3)

Lycka till!

1.a) Characteristic equation for the homogeneous solution is

$$r^2 - 3r + 2 = 0, \quad r_1 = 1, r_2 = 2.$$

The homogeneous solution is therefore

$$a_n^h = c_1 + c_2 2^n.$$

Ansatz for the particular solution (2 is a root of the characteristic equation):

$$a_n^p = cn2^n$$

gives $c = \frac{1}{2}$. Fitting the initial conditions, i.e. adapt the constants c_1, c_2 gives

$$a_n = 2 - 2^n + 2^{n-1}.$$

b) We apply the root criterion to $b_n = \left| \left(\frac{a_n}{a_{n+1}} \right)^n \right|$:

$$\lim_{n \rightarrow \infty} b_n^{1/n} = \lim_{n \rightarrow \infty} \left| \frac{2 - 2^n + n2^{n-1}}{2 - 2^{n+1} + (n+1)2^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n2^{n-1}} - \frac{1}{n} + \frac{1}{2}}{\frac{1}{n2^{n-1}} - \frac{1}{n} + 1} \right| = \frac{1}{2} < 1.$$

Therefore the series converges.

2. We will use Euler's formula and set $w = e^{iz+i\frac{3}{2}\pi}$. We get

$$w + \frac{1}{w} = -4 \quad \text{or} \quad w^2 + 4w + 1 = 0.$$

Hence $w_{1,2} = -2 \pm \sqrt{3}$ och

$$i(z + \frac{3}{2}\pi) = \log(-2 \pm \sqrt{3}) = \ln(2 \mp \sqrt{3}) + (2k+1)\pi i, \quad k \in \mathbb{Z}.$$

Finally,

$$z = -i \ln(2 \mp \sqrt{3}) + (2k - \frac{1}{2})\pi, \quad k \in \mathbb{Z}.$$

b) The radius of convergence equals the distance from the origin (the series is centered at $z = 0$) to the closest singularity, i.e. to the closest solution of the equation from a). We have $\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3})$ and therefore (with $k = 0$) the series converges for

$$z \in \left\{ w \in \mathbb{C} : |w| < \sqrt{\frac{\pi^2}{4} + \ln^2(2 + \sqrt{3})} \right\}.$$

3.a) f is even, so $b_k = 0$. For the exponential series we have

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin t| e^{-ikt} dt = \frac{1}{\pi} \int_0^{\pi} \sin t e^{-ikt} dt.$$

Since $e^{-ik(t-\pi)} = (-1)^k e^{ikt}$ follows $c_k = 0$ for odd k .

$$\begin{aligned} c_{2k} &= \frac{1}{\pi} \int_0^\pi \sin t e^{-2kit} dt = \frac{1}{2\pi i} \int_0^\pi (e^{it} - e^{-it}) e^{-2kit} dt = -\frac{1}{2\pi} \frac{4}{4k^2 - 1} \\ &= -\frac{1}{2\pi} \frac{4}{(2k+1)(2k-1)}. \end{aligned}$$

Using $a_k = c_k + c_{-k}$ (and the continuity of f) we get

$$|\sin t| = -\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{4}{(2k+1)(2k-1)} e^{2kit} = \frac{2}{\pi} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{4}{(2k+1)(2k-1)} \cos(2kt).$$

b) We will use Parseval's formula.

$$\frac{1}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin t|^2 dt = \frac{4}{\pi^2} + \frac{1}{2} \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2(2k+1)^2}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2(2k+1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

4.a) u has to be harmonic, so

$$0 \equiv u''_{xx} + u''_{yy} = (g''(x) - g(x)) \cos y \quad \text{or} \quad g''(x) = g(x).$$

The latter differential equation has the general solution $g(x) = c_1 e^x + c_2 e^{-x}$. We also have $f(0) = 1 + i \implies u(0,0) = 1 \implies g(0) = 1 \implies c_1 + c_2 = 1$. We separate

$$u(x, y) = (c_1 e^x + (1 - c_1) e^{-x}) \cos y = c_1 e^x \cos y + (1 - c_1) e^{-x} \cos y = u_1 + u_2.$$

The harmonic conjugates are $v_1(x, y) = c_1 e^x \sin y + d_1$ and $v_2(x, y) = -(1 - c_1) e^{-x} \sin y + d_2$, $d_{1,2} \in \mathbb{R}$, respectively. Setting $z = x, y = 0$ and using the identity theorem we get

$$f(z) = c_1 e^z + (1 - c_1) e^{-z} + id \quad d \in \mathbb{R}.$$

Finally, $f(0) = 1 + i \implies d = 1$ and $f(i\frac{\pi}{2}) = 2i \implies c_1 = 1$. Hence, $f(z) = e^z + i$.

5.a) We use $\cos(2x) = \Re e(e^{2ix})$ and consider

$$\int_C \frac{e^{2iz}}{1+z^2} dz = \int_{-R}^R \frac{e^{2ix}}{1+x^2} dx + \int_{C_R} \frac{e^{2iz}}{1+z^2} dz$$

where $C_R = \{z \in \mathbb{C} : |z| = R, \Im m(z) > 0\}$. By Jordan's lemma the second integral on the right-hand-side vanishes as $R \rightarrow \infty$. Using the residue theorem ($z = i$ is the only singularity surrounded by C) we get

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^2} dx = \Re e \left(\int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^2} dx \right) = \Re e \left(2\pi i \cdot \text{Res}_{z=i} \left(\frac{e^{2iz}}{1+z^2} \right) \right) = \pi e^{-2}.$$

b) The function $f(z) = \frac{\cos(2z)}{1+z^2}$ has two singularities $z_{1,2} = \pm i$. Let γ be a closed smooth curve. By $W_i(\gamma), W_{-i}(\gamma) \in \mathbb{Z}$ we denote the winding number (with orientation) of γ around i , respectively $-i$. That is we count how often the curve γ encircles each singularity taking into account the direction. The residue theorem says

$$\begin{aligned} \int_{\gamma} f(z) dz &= 2\pi i (W_i(\gamma) \operatorname{Res}_{z=i}(f(z)) + W_{-i}(\gamma) \operatorname{Res}_{z=-i}(f(z))) = W_i(\gamma)\pi \cos(2i) - W_{-i}(\gamma)\pi \cos(2i) \\ &= m\pi \cos(2i), \end{aligned}$$

where $m = m(\gamma) = W_i(\gamma) - W_{-i}(\gamma)$ is an arbitrary integer (depending on the curve γ).

6.a) Let $0 < r < 1$ and $|z| \leq r$ then

$$\sum_{n=1}^{\infty} \left| \frac{z^n}{1-z^n} \right| \leq \sum_{n=1}^{\infty} \frac{r^n}{1-r^n} \leq \frac{1}{1-r} \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} < \infty$$

and the series converges uniformly on $\{z \in \mathbb{C} : |z| \leq r\}$ and hence converges for $|z| < 1$.

If $|z| > 1$ then

$$\lim_{n \rightarrow \infty} \left| \frac{z^n}{1-z^n} \right| \geq \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{|z|^n}} = 1 > 0$$

and the series diverges since the terms do not tend to zero.

If $|z| = 1$ then $|1 - z^n| \leq 2$ and

$$\lim_{n \rightarrow \infty} \left| \frac{z^n}{1-z^n} \right| \geq \frac{1}{2} > 0$$

and again the series does not converge.

To summarize the series diverges for $|z| \geq 1$, converges for $|z| < 1$ (and converges uniformly on each disc $|z| \leq r < 1$).

Remark: The series is not defined for $z = e^{i\frac{p}{q}\pi}$, $p, q \in \mathbb{Z}$.

b) Let $|z| \leq r < 1$. By the view of a) the left-hand-side series converges uniformly and we can change the order of the terms in an arbitrary manner. By using $\frac{1}{1-z^k} = \sum_{l=0}^{\infty} (z^k)^l$ for $|z| < 1$, we conclude

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{z^n}{1-z^n} &= \sum_{k=1}^{\infty} \frac{z^k}{1-z^k} = \sum_{k=1}^{\infty} z^k \left(\sum_{l=0}^{\infty} (z^k)^l \right) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} z^{(l+1)k} \\ &= \sum_{n=1}^{\infty} \sum_{k|n} z^n = \sum_{n=1}^{\infty} \psi(n) z^n. \end{aligned}$$